The Mathematics of Getting to the Moon: A Case Study of Problem Based Learning

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Abstract

An example problem is given. It concerns estimating the time taken for a simplified rocket to hit the moon. The PBL approach described is suited to several categories of student: those studying Ordinary Differential Equations (ODEs); those who have already been taught ODEs and need to know how to solve them on the computer; those who already know about ODEs and wish to study the topic of uncertainty, i.e. the probability that the rocket hits the moon. The research question is "How do the students experience their learning of ODEs in the PBL process?" This paper looks at features of a problem that are supportive of its use in problem-based learning (PBL). As the problem domain under consideration is STEM, some of these features are general ones and apply to any discipline whilst others are specific to STEM disciplines. There are too many features to list in the abstract. The suitability of this example problem for PBL is explained by showing how the example problem satisfies all the features referred to above for a PBL problem. The author has tackled the 'rocket to the moon' problem to the extent that a competent final year undergraduate might tackle it for the submission of a thesis. There are different stages of the rocket's flight. The equation(s) of motion for the rocket at the different stage(s) under consideration must be derived. This is made easier by having an idealized rocket and a simplified model of the Earth, its atmosphere, the Moon and how the Moon orbits the Earth. The Earth's atmosphere is a complex phenomenon, and this is modelled as layers. The layers closest to the Earth offer less resistance to the rocket's motion than those layers further away from the Earth. Students could work individually or in groups. The process of finding *a* solution is as important as finding the *right* solution. The problems involve students in writing computer programs. The languages that are best suited are those that have built-in routines for solving ODEs. The overall problem was solved by the author using Octave, which is freely available software very similar to MATLAB. In the classroom, a tutor would need to support students in debugging programs. The students (and other tutors) should find this PBL approach an acceptable means of learning. Having tackled the ‘rocket to the moon’ problem, the author can recommend it as highly suited to a candidate problem for PBL.

Keywords: Simulating a Rocket Launch; Rocketry; Space Flight; Predicting Probability; Worked Examples

Type of contribution: Research paper.

1. Introduction and Statement of the Problem

The author tackled a problem posed by (Beauzamy, 2016). It is a large problem and requires a considerable amount of time to solve it. Having completed the problem, the author thought that the problem, or reduced forms of the problem, would be suitable for posing to students.

This paper is concerned with trying to shed more light on the problems that are suited to problem-based learning (PBL). It does this by considering a single problem and identifying the features of the problem that seem to make it well-suited to PBL. The selected problem is suitable for students studying Ordinary Differential Equations (ODEs).

This article is an account of how to obtain the flight times of a rocket trajectory reaching a specific target using elementary dynamics. In the current case, it discusses the motion of a rocket targeted at the Moon, firstly where it is flying through the atmosphere with reducing densities (hence reducing drag) and its “free” trajectory towards the moon. In the author’s view it is written at a level that is accessible to the readership of the conference proceedings and is a worthy contribution. The problem involves launching a simplified rocket to the Moon. The model of the scenario is now described.

The first step in creating a model is to break the problem down into separate pieces. For the problem under consideration, the pieces are the rocket, the Earth and Moon, and Earth’s atmosphere. We now have all the parts we need for our model. Each part will be described in turn.

The mass of the rocket is 1 tonne. This mass stays constant; it does not change due to fuel being combusted. The shape of the rocket is a sphere of radius 1 metre. The velocity of the rocket always acts vertically. The Moon’s orbit is assumed to be circular, with radius 384,405 km, and this is in the plane in which the position that the Sun and Earth always lie. The rocket is subject to two gravitational forces, those of the Earth and the Moon. The Earth is spherical with radius

6,378 km and mass 6 X 1024 kg. Similarly, the Moon is spherical with radius 1,738 km and mass 7.3 X 1022 kg. The density of the air at different altitudes is shown in Table 1. The air density is assumed to be constant between two adjacent altitudes shown in the table, e.g.

$$density = \frac{1.22500 + 0.736116}{2}$$ for altitudes between 0 m and 5000 m.

Table 1: Air density at different altitudes.

Altitude (m)	Density (kg/m^3)
0	1.22500
5000	0.736116
…	…
50000	0.000977525

The author does not know whether the model described above is the standard analytical model for solving rocket flight problems.

The formula used to calculate air resistance (drag) is:

$$F_D = \frac{1}{2}\rho V^{\alpha} C_D A$$

For ease of reference, the parameters used in the formula are shown in Table 2.

Table 2: The model's parameters

	Interpretation	Values
F_D	Drag force	
ρ	Density of air	See Table 1
V	Speed of the rocket (m/s)	
α	A coefficient. Its value depends on the object's speed. Our object is travelling at hypersonic velocity.	Between 2 and 2.5
C_D	Drag coefficient	0.5 for a sphere
A	Cross-sectional area of sphere (m^2)	π

2. Limitations of Study

The study described here considers only one problem. Furthermore, the problem has not been exposed to students.

3. Literature Review

In this research, interest focuses on identifying those problems that are suited to PBL. Duch *et al*. (2001) describes the 'Characteristic of Good PBL Problems':

1. "An effective problem must first engage students' interest and motivate them to probe for deeper understanding of the concepts being introduced …"
2. Some problems that work well require students to make decisions. For this kind of problem, students should be asked to justify their decisions.
3. The problem should have a level of complexity such that students would benefit from working in a group.
4. The problem should relate to previously learned knowledge. When introducing the problem, open-ended questions should be posed so that students become interested.
5. The problem should relate to some of the learning outcomes of the course in which the topic is being taught. Studying the problem is an opportunity to build on previous knowledge, learn new concepts and connect new knowledge with concepts in other courses, if not disciplines.

Let us look at point 1 above in more detail. Dewey (1910) states "… the origin of thinking is some perplexity, confusion, or doubt. Thinking is not a case of spontaneous combustion … There is something specific which occasions and evokes it." The Buck Institute of Education (BIE) focuses on project-based learning. Some employees of BIE wrote a book (Larmer *et al*., 2015) in which there is a model entitled 'Gold Standard Project Based Learning.' At the top of the diagram of the model is 'Challenging Problem or Question.' Furthermore, "… the theoretical basis of problem-based learning is closely connected with learning at work" (Poikela & Nummenmaa, 2006).

Let us now consider what is meant by the PBL process. A discussion to illustrate "the possible use of PBL in a core module of a Postgraduate Diploma in Education Programme entitled 'The Psychology of Pupil Development and the Learning Process'" is given by Tan (2002). In it there is a figure that "provides a schema of the PBL process that trainee teachers go through in their five tutorials. More generally, the figure can be amended slightly to give Fig. 1, which gives a breakdown of the PBL process.

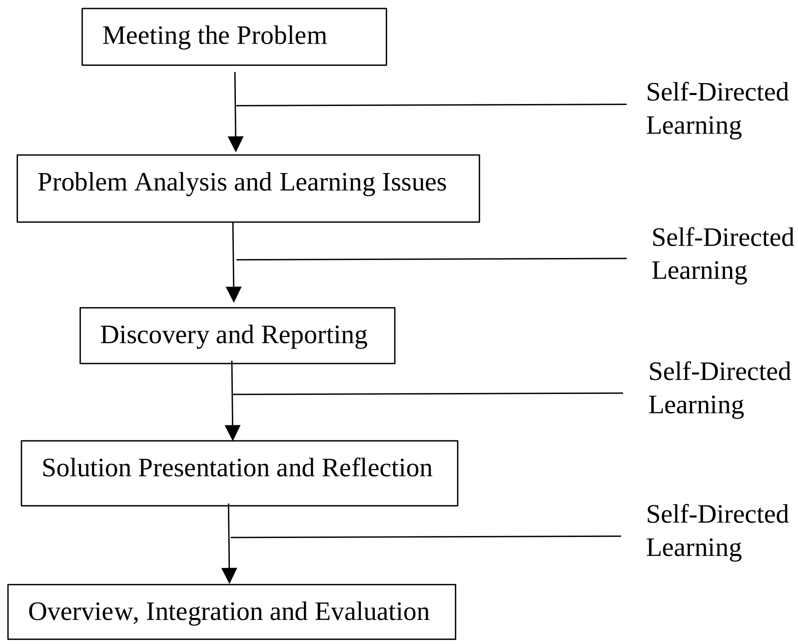

Figure 1: The PBL process (based on Tan (2002)).

The research described here concerns the study of ODEs. Following are some efforts at applying problem-based learning to ODEs. Ventura-Medina *et al*. (2006) describe a fourth-year Mathematics module involving partial differential equations and techniques for their solution. The techniques all entail reducing the partial differential equations to ordinary differential equations. "Students work in small groups using Problem-Based Learning (PBL) to specialize in a particular part of the course syllabus. They then taught their specialist part to their peers, and designed a suitable assessment by which their peers' learning was gauged."

Lewis & Powell (2016) use PBL in college math classes. One project is centred around a Humans vs. Zombies game. This project is used in an undergraduate differential equations course. Mora *et al*. (2002) introduce ConsMath (CONStraint-based MATH teaching), a computer system enabling the teacher to creative interactive Mathematics documents. "It has been tested by Mathematics teachers in order to build practicing materials on Calculus and

Ordinary Differential Equations, and the use of the corresponding sets of problems by students has started recently."

Dian & Apriani (2019) use PBL to teach ODEs. They subdivide the learning activity into six stages: problem introduction; search for analytic solutions; introduction of the approach methods; search for the approach solutions manually; search for the solutions using computer help; the meaning of solutions. Meanwhile, Deters *et al*. (10) look at teaching ODEs using problem-based learning.

4. Identifying Features of a Problem that are Supportive of PBL

Some features of a problem that are supportive of PBL:

- the subject matter supports the curriculum
- the problem is topical
- it is easy for learners to see how the subject matter is useful in the real world
- students can be asked to identify and discuss any strong assumptions that have been made in the problem
- for STEM disciplines: solving the problem requires students to use ICT, if not be engaged in some computer programming
- the problem can be broken down into 'chunks.' Students can be asked to tackle one chunk to introduce the problem.
- students can tackle one, many, or all chunks. This means that the problem can be an individual assessment or a group assessment.
- the extent of the problem is such that it can be given as a final year project.
- after tackling a chunk, students should have a rough idea of whether their solution is sensible or not, with reference to the real world.
- for STEM disciplines: the problem should be suited to being solved using a 4th generation computer language, such as Matlab or Octave, rather than having to resort to the intricacies of a language such as Java (unless, of course, the students are computing undergraduates).

- the subject matter should have a degree of uncertainty, as large real-world problems are not deterministic. There are two aspects to uncertainty:
    - Sensitivity analysis
    - Calculating the probability of an event

4.1 The subject matter supports the curriculum

The example problem is suited to three types of courses:

- a course where ODEs are studied;
- a course where the study of ODEs is a prerequisite and where the curriculum involves their solution on the computer;
- a course where the study of ODEs is a prerequisite and where the curriculum involves the study the topic of uncertainty.

4.2 Strong assumptions that have been made in the problem

It is very well-known that rockets are designed with minimum weight as a key constraint. Therefore, any fuel associated with the provision of thrust should really be accounted in some way. In the current case, it would be likely that the mass of the fuel at launch, which is consumed by the end of the thrust phase, would be a significant proportion of the total mass of the rocket, the latter being assumed to be only 1000 kg. Hence, mass would be a significant function of time in the thrust phase – this could quite easily be incorporated in the equation of motion (probably not so easy to solve analytically though). Nevertheless, for the current problem, the fact that mass is assumed to be constant is a strong assumption and deserves some discussion. Another point on which it is worth elaborating is the fact that the rocket is assumed to be spherical. However, most launch vehicles tend to be long and slender that resemble a bullet, as opposed to a cannonball. How the differing geometry affects the drag coefficient should be discussed.

An example discussion of how the two points above might proceed is now given. Students have been asked to tackle the problem posed earlier. Two of the characteristics specified in the problem are worth elaborating on – the mass ratio and the shape of the rocket. Consider the following formula:

$$\text{Mass ratio} = \frac{\text{Vehicle mass} + \text{propellant mass}}{\text{Vehicle mass}}$$

For a Boeing 747 the mass ratio is 2, for the X-15 rocket-powered aircraft it was 2.3, for the V-2 it was 3.85. However, for the Ariane 5 it is 39.9. In the problem being tackled, the mass is assumed to be invariant, for simplicity; that is, the mass ratio is 1. As regards the shape of the rocket, it determines the drag coefficient and, hence, the amount of drag acting on the rocket. Long thin shapes with a pointed top, such as the shape of a bullet, have a very low drag coefficient. The problem posed states that the rocket is spherical. A sphere has a drag coefficient of 0.5, much larger than that of a traditional rocket shape**.**

4.3 The problem can be broken down into 'chunks'

The author has identified three stages to solving the problem. Stage 1 is the equation of motion for the rocket between the Earth's surface and 50,000 m above the surface. Stage 2 is the equation of motion for the rocket between 50,000 m above the Earth's surface and the height when the thrust runs out. Stage 3 is the equation of motion for the rocket between the height when the thrust runs out and the Moon's surface. To solve the problem, one must first formulate the ODEs and then numerically solve them using Octave, for example. To formulate the ODEs, there are seven 'chunks' that can be tackled: Stage 1: initial part; Stage 1: middle part with $\alpha = 2$; Stage 1: middle part for any value of α; Stage 1: end part; Stage 2; Stage 3; Stage 1b: For very large values of α, the thrust runs out before the rocket reaches 50,000 m

One or more of these chunks can be given to students. For example, let us look at what we require students to do for Stage 1b. It is required to derive the equation of motion for the rocket between the height when the thrust runs out and 50,000 m above the surface. Students could be asked to first draw a diagram to show the forces acting on the rocket. They should come up with something like Figure 2. The forces acting on the rocket are Earth's gravity and drag. The Earth is hundreds of millions of metres from the Moon and so the Moon's gravity can be ignored.

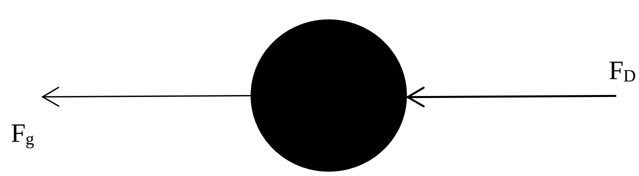

Figure 2: Forces acting on the rocket in Stage 1b.

Next, students could be asked to derive the equations of motion. An example working follows.

The sum of the forces acting on the rocket is:

$$\sum F = -mg - DV^{\alpha} = m\frac{dV}{dt}$$
$$\frac{dV}{dt} = -g - \frac{D}{m}V^{\alpha}$$
$$\frac{dy}{dt} = V$$

The second type of exercise that could be given to students is to numerically solve the ODEs that they have found. For example, with Stage 1b students would have to first write Octave code to solve the ODEs. Students could be told the height that the rocket reached when the fuel ran out and the speed that the rocket was travelling. They could then be asked to work out how high the rocket travels before it starts to return to Earth. This would involve students numerically solving the ODEs for complete layer(s) and partial layer(s) of the atmosphere.

5. Conclusion

An attempt has been made to identify the features of a problem that are supportive of PBL. To illustrate what each of these features means, an example problem on rocket flight has been used. Some of the features are general in nature whilst others are suited to STEM disciplines. When a tutor is considering using a particular problem for PBL, the tutor can study the problem to determine which features it possesses. On the basis of this, the tutor will be in a better position to gauge the suitability of the problem for use in PBL.

Octave is introduced to students as it is a suitable environment for mathematics-based problems. To use Octave students are required to master basic programming algorithms. There is no need to learn about repetition algorithms for the rocket problem; the solutions of the different stages of the rocket's flight can be pieced together manually. We can see that Octave can be used with problem-based learning as an alternative to other forms of learning.

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